

An Introduction to Generalized Yang-Mills Theories

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Abstract

In generalized Yang-Mills theories scalar fields can be gauged just as vector fields in a usual Yang-Mills theory, albeit it is done in the spinorial representation. The presentation of these theories is aesthetic in the following sense: A physical theory using Yang-Mills theories requires several terms and irreducible representations, but with generalized Yang-Mills theories, only two terms and two irreducible representations are required. These theories are constructed based upon the maximal subgroups of the gauge Lie group. The two terms of the lagrangian are the kinetic energy of fermions and of bosons. A brief review of Yang-Mills theories and covariant derivatives is given, then generalized Yang-Mills theories are defined through a generalization of the covariant derivative. Two examples are given, one pertaining the Glashow-Weinberg-Salam model and another $SU(5)$ grand unification. The first is based upon a $U(3) \supset U(1) \times U(1) \times SU(2)$ generalized Yang-Mills theory, and the second upon a $SU(6) \supset U(1) \times SU(5)$ theory. The possibility of expressing generalized Yang-Mills theories using a five-dimensional formalism is also studied. The situation is unclear in this case. At the end a list of comments and criticisms is given.

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1 Introduction.

Yang-Mills theories have enjoyed tremendous success in the understanding and quantization of three of the fundamental forces of nature: the electromagnetic, weak and strong forces. The discovery and application of this powerful idea was certainly one

of the great scientific achievements of the twentieth century. Their application to the field of elementary particles has been so successful it has become the Standard Model of the field. Through these theories the weak and electromagnetic force have been unified into a renormalizable, phenomenologically correct theory based on the $SU(2) \times U(1)$ Lie group. Quantum chromodynamics, based on $SU(3)$, has become by far the most promising theory we have to understand the nonperturbative complexities of the strong force. Finally, the efforts that have been done at grand unification, although unsatisfactory in last analysis, have been successful enough to show that, whatever form the final theory we all wait for may have, at low energies it is going to look very much like a Yang-Mills theory.

In spite of all these achievements there is a very understandable attitude of dissatisfaction in the overall status of the Standard Model. The large number of empirical parameters, the arbitrary structure of the Higgs sector, the rather *ad hoc* multiplicity of terms and irreducible representations (irreps), plus several puzzling situations it presents, such as the hierarchy and generation problems, are all unpleasant reminders that we need a more powerful and general theory to be able satisfactorily understand the workings of the universe.

Here we give an introduction to Generalized Yang-Mills Theories (GYMTs), which, as their name implies, are a generalization of the usual Yang-Mills theories (YMTs). As it is the case in YMTs, a local gauge symmetry is enforced upon the lagrangian; however, the role of gauge fields is not taken only by vector fields, but also by scalar ones. The resulting theory is still gauge invariant, but it allows the Higgs fields of quantum field theories to be included as part of the covariant derivative.

These theories have not been properly structured yet as mathematical theories. Their development till now has been from a practical point of view, keeping in mind only their empirical applicability to the theories of high energy physics. They have been applied successfully to two particular examples: the Glashow-Weinberg-Salam Model (GWSM), and a case of grand unification. In both cases the Higgs fields have been incorporated into the covariant derivative, the many terms of each of these theories reduced to only two, and the large number of irreps of each of these theories reduced to only two: the irrep of the fermions and the irrep of the bosons (always the adjoint).

For each Lie group and one of its maximal subgroups there seems to be a GYMT. The GWSM's GYMT is based upon $SU(3) \supset SU(2) \times U(1)$, and $SU(5)$ grand unification upon $SU(6) \supset SU(5) \times U(1)$.

We shall be using the metric

$$(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1), \quad (1)$$

and the Dirac matrices γ^μ , $\mu = 0, 1, 2, 3$, obeying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

We use a representation in which $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$, $i = 1, 2, 3$. We also use the chirality matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, so that $\gamma^{5\dagger} = \gamma^5$.

2 Brief review of Yang-Mills theories.

The notion of gauge invariance goes back to Weyl¹. Consider the quantum electrodynamics lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial - eA)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (2)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This lagrangian has a straightforward gauge invariance in the sense that the Maxwell tensor $F_{\mu\nu}$, is invariant under the substitution $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$. The electron wavefunction is invariant under a global (that is, spacetime independent) transformation $\psi \rightarrow e^{i\delta}\psi$. To make it invariant under a local transformation we must perform a simultaneous transformation on the electromagnetic potential field A_μ and the electron wavefunction ψ , otherwise there would be an extra term left from the application of Leibnitz's differentiation rule to the product $e^{i\delta}\psi$.

From a mathematical point of view the electromagnetic potential is a connection dictating how vector fields are going to displace when they move along specific paths along the base manifold R^{3+1} . Its gauge invariance allows for the establishment of a principal vector bundle structure, with different connections, that is, electromagnetic potentials, over each open set of the covering of the base space. In the overlapping zones of two open sets the respective connections must only differ by a gauge transformation $\partial_\mu \Lambda$. This conditions ensures that parallel transport develops smoothly as any path is traversed along the base manifold. It is interesting that from a mathematical point of view gauge freedom has the advantage of allowing topologically nontrivial vector bundles, while from a physical point of view the main advantage of gauge freedom is that it allows for an improvement of the divergence of momentum integrals in quantum loop calculations.

2.1 An abelian Yang-Mills theory.

Let $U = e^{-i\alpha(x)}$ be an element of a local transformation based on the $U(1)$ Lie group, so that the electrically charged fermion field transforms as $\psi \rightarrow U\psi$. The electromagnetic potential is required to obey the gauge transformation law

$$A_\mu \rightarrow A_\mu + e^{-1}(\partial_\mu \alpha), \quad (3)$$

so that, defining

$$D_\mu \equiv \partial_\mu + ieA_\mu, \quad (4)$$

the covariant derivative must transform as

$$D_\mu \rightarrow UD_\mu U^{-1}. \quad (5)$$

In this equation the derivative that is part of the covariant derivative is acting indefinitely to the right and not only on the U^{-1} . We shall call such operators *unrestrained*,

while the ones that act only on the immediately following object, such as the partial in (3), we shall call *restrained*, and use a parenthesis to emphasize that the action of the differentiation operator does not extend any further to the right. Note that while the operators D_μ and $D_\mu D_\nu$ are unrestrained, the operator $[D_\mu, D_\nu]$ is restrained, even though it *looks* unrestrained.

Let us verify it really is restrained. Let $f = f(x)$ be a twice differentiable function. Then

$$\begin{aligned} -ie^{-1}[D_\mu, D_\nu]f &= \partial_\mu A_\nu f - A_\nu \partial_\mu f - \partial_\nu A_\mu f + A_\mu \partial_\nu f \\ &= ((\partial_\mu A_\nu) - (\partial_\nu A_\mu)) f, \end{aligned} \quad (6)$$

an expression that has no unrestrained derivatives. There are mathematical and physical reasons why the curvature, which, as we shall see, is what the expression $[D_\mu, D_\nu]$ represents, should contain only restrained differential operators. In any case the point we want to make here is that in GYMT all expressions of physical significance always automatically turn out to have restrained operators.

Let us rewrite the lagrangian using the covariant derivative:

$$\mathcal{L}_{QED} = \bar{\psi} i \not{D} \psi + \frac{1}{4e^2} [D_\mu, D_\nu] [D^\mu, D^\nu]. \quad (7)$$

The advantages with this formulation are twofold: on one hand the mathematical meaning is transparent, and on the other the proof of gauge invariance becomes straightforward. Thus, the effect of the gauge transformation on $\bar{\psi} \not{D} \psi$ results in $\bar{\psi} \not{D} U^{-1} U \psi$, but, since the derivative is acting on all that follows to its right, the unitary operators cancel and invariance follows. To verify the invariance of the second term on the right of (7) notice that under the gauge transformation (5),

$$[D_\mu, D_\nu] [D^\mu, D^\nu] \rightarrow U [D_\mu, D_\nu] U^{-1} U [D^\mu, D^\nu] U^{-1}.$$

Since we have already verified that the derivatives in the commutators of D 's do not act on anything that comes after them, the unitary operators can be commuted with the commutators and they cancel, leaving the original expression.

2.2 A nonabelian Yang-Mills theory.

The gauge structure of electromagnetism was generalized by Yang and Mills². The fermions transform as before $\psi \rightarrow U\psi$, but in their generalization of electromagnetism the U do not commute among themselves. The gauge fields are contracted with matrices that are a representation of the adjoint representation of simple Lie groups. The dimensionality of the adjoint representation is equal to the number of generators of the Lie group. One associates to each generator one vector field, and the sum of the products of each generator times a vector field constitutes the connection.

To illustrate a nonabelian YMT let us take the Lie Group to be $SU(N)$. Its fundamental representation \mathbf{N} is of dimension N , and its adjoint \mathbf{Adj} is of dimension

$N^2 - 1$. Let A_μ^a , $a = 1, \dots, N^2 - 1$, be the fields to be associated with the generators. We take the fermions to be in the fundamental representation \mathbf{N} , so that the wavefunction $\psi(x)$ has N spinorial components. However, in order to construct the lagrangian, we do not work with the $(N^2 - 1) \times (N^2 - 1)$ matrices that constitute the usual representation of the adjoint. The way to proceed instead is to use the fact that for $SU(N)$, $\mathbf{N} \times \mathbf{N}^* = \mathbf{Adj} + \mathbf{1}$. This implies that there must exist matrices T^a of size $N \times N$ that are the Clebsch-Gordan coefficients generators relating \mathbf{N} , \mathbf{N}^* and \mathbf{Adj} . As a matter of fact these Clebsch-Gordan coefficients in this case are no other than the generators in the fundamental representation. Thus for these groups $A_\mu^a T^a$ is a $SU(N)$ tensor transforming as

$$A_\mu^a T^a \rightarrow U A_\mu^a T^a U^{-1}, \quad (8)$$

where U is an element of the fundamental representation \mathbf{N} . The A_μ^a , seen as an $N^2 - 1$ component vector, transforms in (8) as the \mathbf{Adj} representation. The expression $\bar{\psi} \gamma^\mu A_\mu^a T^a \psi$ is obviously gauge-invariant.

To assure gauge invariance in the presence of the fermion kinetic energy derivative we must introduce a covariant derivative again. Thus we define

$$D_\mu = \partial_\mu + A_\mu, \quad A_\mu = ig A_\mu^a(x) T^a \quad (9)$$

where g is a coupling constant, and assign the following gauge transformation to this dynamic field:

$$A_\mu \rightarrow U A_\mu U^{-1} - (\partial_\mu U) U^{-1}. \quad (10)$$

This in turn insures that the covariant derivative transforms similarly to the way it did in the abelian case, that is, as in (5).

The lagrangian of the YMT is then:

$$\begin{aligned} \mathcal{L}_{YMT} &= \bar{\psi} i \not{D} \psi + \frac{1}{2g^2} \widetilde{\text{Tr}} ([D_\mu, D_\nu] [D^\mu, D^\nu]) \\ &= \bar{\psi} i (\not{\partial} + \not{A}) \psi + \frac{1}{2g^2} \widetilde{\text{Tr}} (\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu])^2, \end{aligned} \quad (11)$$

where the trace is over the $SU(N)$ group generators and we follow the usual normalization condition

$$\widetilde{\text{Tr}} T^a T^b = \frac{1}{2} \delta_{ab}. \quad (12)$$

A tilde is used over the trace to differentiate it from the trace over Dirac matrices that we shall also use. We are missing in this lagrangian scalar boson Higgs fields. With them it is possible to construct other terms in the Lagrangian that are also simultaneously Lorentz- and gauge-invariant and are of the general form fermion \times Higgs \times fermion. These, called the Yukawa terms, are of extreme importance since they constitute the natural channel these theories have to generate fermion mass. This comes about through a hypothesized nonzero vacuum expected value (VEV) of the Higgs. The lagrangian must also contain the kinetic energy of the Higgs, that has the form $|D_\mu \varphi|^2$. Furthermore, to justify the nonzero VEV for the Higgs another term is usually added, a potential $V(\varphi)$ that has a local minimum at some constant and uniform value $\varphi = v$.

3 The gauge field as a connection.

The local antihermitian matrix function defined in (9) can serve as a connection. Let $\phi(x)$ be an element at point x of an N dimensional vector space. Then, for an infinitesimal path dx in R^{1+3} we *define* the parallel displacement of ϕ to be

$$d\phi_{||} \equiv dx^\mu A_\mu \phi. \quad (13)$$

Notice that the vector does not change length in a YMT due to its parallel displacement. The change in ϕ 's length is given by

$$(\phi + d\phi_{||})^\dagger (\phi + d\phi_{||}) - \phi^\dagger \phi \approx dx^\mu \phi^\dagger (A_\mu + A_\mu^\dagger) \phi = 0$$

due to the antihermitian nature of A_μ .

3.1 An unitary operator as functional of a path.

Consider now a path C in R^{1+3} from x_1 to x_2 parametrized with s , $0 \leq s \leq 1$, so that x is a function $x(s)$ and $x(0) = x_1$, $x(1) = x_2$. Since parallel displacement does not change the length of the field ϕ its effect can be simply expressed as multiplication by a unitary operator $U(x(s))$. The change in the vector at any particular point s in C is given by $dU\phi = dx^\mu A_\mu U\phi$, so the unitary operator has to satisfy the equation

$$\frac{dU}{ds} = \frac{dx^\mu}{ds} A_\mu U. \quad (14)$$

Its solution is

$$U(s) = Pe^{I(s)}, \quad I = \int_0^s ds \frac{dx^\mu}{ds} A_\mu, \quad (15)$$

where P is the path ordering operator, which arranges operators so that they are placed according to the value of s in their argument, from higher values on the left to lower on the right. Let us prove that (15) is really the solution of the differential equation. The following is a simple demonstration that does not involve changing the integration limits:

$$\begin{aligned} \frac{dU}{ds} &= \frac{d}{ds} P \left(1 + I + \frac{1}{2} I^2 + \frac{1}{3!} I^3 + \dots \right) \\ &= \dot{I} + \frac{1}{2} P(\dot{I}I + I\dot{I}) + \frac{1}{3!} P(\dot{I}II + I\dot{I}I + III) + \dots \\ &= \dot{I}P \left(1 + I + \frac{1}{2} I^2 + \dots \right) = \frac{dx^\mu}{ds} A_\mu U. \end{aligned} \quad (16)$$

The path ordering operator P does not affect the unitarity of the $U(s)$. One can show that $U^\dagger = U^{-1}$ simply by expanding the series and performing the required algebra. This is consistent with our previous result that parallel transport cannot change the length of a vector.

3.2 Group generated by integration over closed paths.

Consider the set $\{C_Q\}$ of all continuous closed paths in spacetime that pass through the point Q , and let $U(C_Q) = P \exp \oint_{C_Q} dx^\mu A_\mu$. Then the set $G_Q = \{U(C_Q)\}$ of unitary operators for all such paths forms a group. The identity corresponds to a path made up of the single point Q . The inverse element to an operator is one with the same integration path but traversed instead in the inverse direction. The path associated with the product of two operators with paths C_Q and C'_Q is one that has a single path $C''_Q = C_Q + C'_Q$ defined as follows: it begins at Q and follows C_Q till it comes back to Q , then joins C'_Q and follows it till it comes back to Q , and then joins C_Q again, forming a closed curve. The closure of the product of two operators hinges precisely on the presence of the path operator P . Thus let two operators be $e^{I(C_Q)}$ and $e^{I(C'_Q)}$, so that their product should then be

$$e^{I(C_Q)} e^{I(C'_Q)} = e^{I(C''_Q)}.$$

But clearly this is going to be true only if $P[I(C_Q), I(C'_Q)] = 0$. One effect of the path ordering operator is to order the integrands in way that is unconnected to their original order, so that this commutator has to be zero.

3.3 Example of the meaning of curvature taken from Riemannian spaces.

Let us introduce curvature in the context of Riemannian geometry. It is well-known that geodesic deviation and parallel transport around an infinitesimally small closed curve in a Riemann manifold are two aspects of the same construction.³ Consider a four dimensional Riemann manifold with metric $g_{\mu\nu}$ and a vector B^μ whose parallel transport around a closed curve C is given by

$$\Delta B_\alpha = \oint_C \frac{dx^\nu}{ds} \Gamma^\sigma{}_{\nu\alpha} B_\sigma ds, \quad (17)$$

where s is a parametrization of the curve and $\Gamma^\sigma{}_{\nu\alpha}$ is the connection that acts on the vectors of the tangent vector space to the Riemann manifold, the Christoffel symbol. For simplicity we take C be a small parallelogram made up of the two position vectors x^μ and y^μ , so that its area two-form is $\Delta S^{\mu\nu} \equiv \frac{1}{2}x^{[\mu}y^{\nu]}$. To perform the integral we first make, about the center of the parallelogram, a Taylor expansion

$$\tilde{\Gamma}^\sigma{}_{\nu\alpha} = \Gamma^\sigma{}_{\nu\alpha} + \Gamma^\sigma{}_{\nu\alpha,\mu} \Delta x^\mu \quad (18)$$

for the Christoffel symbol and a parallel displacement

$$\tilde{B}_\sigma = B_\sigma + \Delta x^\mu \Gamma^\tau{}_{\mu\sigma} B_\tau \quad (19)$$

for the vector, both ending at some point separated a distance Δx^μ from the center. The quantities with tilde will take their values at each of the four sides of the parallelogram, which we shall call #1, #2, #3 and #4, counting counterclockwise. The

contribution of side #1 to the integral is:

$$\Delta y^\nu \tilde{\Gamma}^\sigma_{\nu\alpha} \tilde{B}_\sigma|_1 = y^\nu (\Gamma^\sigma_{\nu\alpha} + \Gamma^\sigma_{\nu\alpha,\mu} x^\mu/2) (B_\sigma + x^\mu \Gamma^\tau_{\mu\sigma} B_\tau/2).$$

The contribution of side #3 is

$$\Delta y^\nu \tilde{\Gamma}^\sigma_{\nu\alpha} \tilde{B}_\sigma|_3 = -y^\nu (\Gamma^\sigma_{\nu\alpha} - \Gamma^\sigma_{\nu\alpha,\mu} x^\mu/2) (B_\sigma - x^\mu \Gamma^\tau_{\mu\sigma} B_\tau/2).$$

The contribution of side #1 and its opposite side #3 is

$$\Delta y^\nu \tilde{\Gamma}^\sigma_{\nu\alpha} \tilde{B}_\sigma|_{1+3} = x^\mu y^\nu (\Gamma^\sigma_{\nu\alpha} \Gamma^\tau_{\mu\sigma} + \Gamma^\tau_{\nu\alpha,\mu}) B_\tau,$$

up to first order. The contribution of side #2 is:

$$\Delta x^\mu \tilde{\Gamma}^\sigma_{\mu\alpha} \tilde{B}_\sigma|_2 = -x^\mu (\Gamma^\sigma_{\mu\alpha} + \Gamma^\sigma_{\mu\alpha,\nu} y^\nu/2) (B_\sigma + y^\nu \Gamma^\tau_{\nu\sigma} B_\tau/2),$$

and that of its opposite side #4 is

$$\Delta x^\mu \tilde{\Gamma}^\sigma_{\mu\alpha} \tilde{B}_\sigma|_4 = x^\mu (\Gamma^\sigma_{\mu\alpha} - \Gamma^\sigma_{\mu\alpha,\nu} y^\nu/2) (B_\sigma - y^\nu \Gamma^\tau_{\nu\sigma} B_\tau/2),$$

so that

$$\Delta x^\mu \tilde{\Gamma}^\sigma_{\mu\alpha} \tilde{B}_\sigma|_{2+4} = -x^\mu y^\nu (\Gamma^\sigma_{\mu\alpha} \Gamma^\tau_{\nu\sigma} + \Gamma^\tau_{\mu\alpha,\nu}) B_\tau.$$

Summing over the four sides one obtains for the change of the vector as it is parallel-transported around the parallelogram:

$$\begin{aligned} \Delta B_\alpha &\equiv \oint_C \Delta x^\mu \tilde{\Gamma}^\sigma_{\mu\alpha} \tilde{B}_\sigma \\ &= x^\mu y^\nu (\Gamma^\tau_{\nu\alpha,\mu} - \Gamma^\tau_{\mu\alpha,\nu} + \Gamma^\sigma_{\nu\alpha} \Gamma^\tau_{\mu\sigma} - \Gamma^\sigma_{\mu\alpha} \Gamma^\tau_{\nu\sigma}) B_\tau \\ &= x^\mu y^\nu R^\tau_{\alpha\mu\nu} B_\tau = \Delta S^{\mu\nu} R^\tau_{\alpha\mu\nu} B_\tau, \end{aligned} \quad (20)$$

where $R^\tau_{\alpha\mu\nu}$ is the Riemann curvature tensor and we have used the antisymmetry of its last two indices. Thus the change in B_σ is proportional to the surface enclosed by the path. If the Riemann tensor is zero, the change in B_σ is null.

For convenience we list some properties this tensor satisfies:

$$\begin{aligned} R^\tau_{\alpha\mu\nu} &= -R^\tau_{\alpha\nu\mu}, & R^\tau_{[\alpha\mu\nu]} &= 0, \\ R^\tau_{\alpha[\mu\nu;\omega]} &= 0, & R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu}, \\ R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta}, & R_{[\alpha\beta\mu\nu]} &= 0. \end{aligned}$$

3.4 Riemann curvature tensor as a commutator of covariant derivatives.

We wish to write the curvature tensor as a commutator of covariant derivatives, as we did with YMT. We are going to prove it in a way that emphasizes the similarity between YMTs and the Theory of General Relativity (TGR), because of the suggestiveness of the exercise.

The covariant derivative of a vector B_σ is given by

$$B_{\sigma;\mu} = \partial_\mu B_\sigma - \Gamma^\tau{}_{\mu\sigma} B_\tau = (\partial_\mu \delta^\tau{}_\sigma - \Gamma^\tau{}_{\mu\sigma}) B_\tau. \quad (21)$$

This is the standard formalism used in the TGR, and uses no implicit indices at all; all are explicit. In YMTs the indices corresponding to internal space, or, as mathematicians say it, the indices of the matrices that make up the transition group and the connection, are left implicit with a remarkable simplification resulting. Let us write (21) leaving implicit the indices that have to do with the tangential vector space. We assume that $\delta^\tau{}_\sigma$ is the identity matrix $\mathbf{1}$ with implicit subindices, and that $\Gamma^\tau{}_{\mu\sigma}$ is one of a set of four matrices A_μ also with implicit subindices. Then

$$\delta^\tau{}_\sigma \rightarrow \mathbf{1}_{\sigma\tau}, \quad \Gamma^\tau{}_{\mu\sigma} \rightarrow (A_\mu)_{\sigma\tau},$$

and the covariant derivative of (21) can be written

$$B_{\sigma;\mu} \rightarrow (\mathbf{1}\partial_\mu - A_\mu)B,$$

where the index of the vector B_σ is, naturally, also kept implicit. In what follows we also omit the identity matrix $\mathbf{1}$, as is almost always done in similar occasions in physics. We can now achieve our goal very efficiently. Notice, too, how unrestrained differentials never appear in the TGR. The previous expression for the Riemann curvature tensor can be written in the interesting form

$$[D_\mu, D_\nu] = \partial_{[\nu} A_{\mu]} + [A_\mu, A_\nu].$$

To prove this we re-introduce the indices and usual names of the variables in the commutator that appears in above's equation. It immediately follows that

$$\begin{aligned} [D_\mu, D_\nu]_{\sigma\tau} &= \Gamma^\tau{}_{\mu\sigma,\nu} - \Gamma^\tau{}_{\nu\sigma,\mu} + \Gamma^\beta{}_{\mu\sigma}\Gamma^\tau{}_{\nu\beta} - \Gamma^\beta{}_{\nu\sigma}\Gamma^\tau{}_{\mu\beta} \\ &= -R^\tau{}_{\sigma\mu\nu}, \end{aligned}$$

that is, the commutator of the covariant derivative is minus the Riemann curvature tensor.

The expressions in lagrangians (7) and (11) contain terms that strongly resemble the curvature expression of Riemannian spaces. As a matter of fact they are the curvatures associated with the connections that appear in principal vector bundles. Our interest here again is not to review general physical and mathematical results, but to show the similarity of the mathematical expressions in two different but equally fundamental physical theories. There is one rather striking difference between these two theories: in the YMT the curvature enters squared in the lagrangian, while in the TGR it enters linear.

3.5 Curvature in a Yang-Mills theory.

We justify now our having identified $[D_\mu, D_\nu]$ with the curvature. Here D_μ is the Yang-Mills covariant derivative defined in (9). We proceed similarly to the way we

did in the Riemannian case. We define the change in a vector χ to be given by the parallel transport of the vector around a closed path C . We maintain the same previous base manifold: four-dimensional spacetime with the flat metric (1). The excess of the vector is:

$$\Delta\chi = \oint \frac{dx^\nu}{ds} A_\nu \chi ds,$$

where s is a parametrization of the curve and A_ν is the Yang-Mills connection. We can now mimic the derivation of the calculation we did of parallel transport for in a Riemannian manifold with connection, and obtain

$$\begin{aligned}\Delta\chi &= x^\mu y^\nu (A_{\nu,\mu} - A_{\mu,\nu} + A_\nu A_\mu - A_\mu A_\nu) \chi \\ &= x^\mu y^\nu [D_\mu, D_\nu] \chi \equiv \Delta S^{\mu\nu} F_{\mu\nu} \chi.\end{aligned}$$

We conclude that curvature in a YMT is given by

$$F_{\mu\nu} = [D_\mu, D_\nu].$$

4 Introduction to generalized Yang-Mills theories.

The sense in which one generalizes a YMT is by promoting scalar fields to the level of gauge fields. In the standard YMT, gauge invariance is assured by demanding the vector gauge field transform as $A_\mu \rightarrow U A_\mu U^{-1} - (\partial_\mu U)U^{-1}$. A scalar gauge field does not have a vector index to associate with the term $(\partial_\mu U)U^{-1}$ in the previous equation. One way to include a one scalar field in a covariant derivative would be to increase the dimension of the base manifold by one, so that $\mu = 0, 1, 2, 3, 4$. However, there are two serious problems with this approach. First, this would not be a true generalization of YMTs, but simply a YMT constructed in a higher dimensionality. Second, the resulting theory is phenomenologically unacceptable: it does predict the physics of the standard model.

The way we proceed to be able to include scalar bosons as gauge fields

is to write a spinorial covariant derivative, and associate the scalar fields with the chiral operator γ^5 . This procedure is very successful from a phenomenological point of view. As usual the gauge bosons are placed in the adjoint representation of a Lie group. The idea is that *all* scalar bosons have to enter the model this way. In this aspect, as in several others, GYMTs are far more restrictive than the usual YMTs. The total number of bosons, vector plus scalar, has to equal to the number of group generators, since they have to be multiplied by the same $N^2 - 1$ matrices T^a as we saw in the YMTs. Fermions fields are placed in irreducible representations (irreps) of mixed chirality according to rules given by the particular choice of maximal subgroup of the original Lie group of the GYMT.

This setup presents us with a difficulty, since the kinetic energy of the vector bosons is not written using a spinorial covariant derivative, so we cannot use the generalized form of the covariant derivative we are planning to implement. We address this difficulty next.

4.1 Gauge field's kinetic energy using a spinorial covariant derivative: abelian case.

There is a 2×1 homomorphism between the Dirac spinorial and vector representations of the Lorentz group. If S is a matrix that represents an element of the spinorial representation, and if ψ is a spinor and \mathcal{A} a slashed 4-vector, then under a Lorentz transformation

$$\psi \rightarrow S\psi, \quad \mathcal{A} \rightarrow S\mathcal{A}S^{-1}.$$

The following identity relates the vectorial formulation of the Yang-Mills kinetic energy with the Dirac spinorial one.

Theorem, abelian case. *Let $D_\mu = \partial_\mu + B_\mu$, where B_μ is a vector field. Then:*

$$((\partial_\mu B_\nu) - (\partial_\nu B_\mu)) ((\partial^\mu B^\nu) - (\partial^\nu B^\mu)) = \frac{1}{8} \text{Tr}^2 \not{D}^2 - \frac{1}{2} \text{Tr} \not{D}^4, \quad (22)$$

where the traces are to be taken over the Dirac matrices.

The differentiations on the right of the equation are not restrained. To prove the theorem it is convenient to use the differential operator

$$O \equiv \partial^2 + 2B \cdot \partial + B^2.$$

It is important to notice that it does not contain any contractions with Dirac matrices, so that when it appears in traces of Dirac matrices it facilitates the calculation. For example, $\text{Tr} O = 4O$, $\text{Tr} O (\not{\partial} \not{B}) = 4O (\partial \cdot B)$, etc. For convenience we give a list of Dirac trace formulae:

$$\begin{aligned} \text{Tr } 1 &= 4, \\ \text{Tr } \not{\phi}_1 \not{\phi}_2 &= 4a_1 \cdot a_2, \\ \text{Tr } \not{\phi}_1 \not{\phi}_2 \cdots \not{\phi}_n &= 0, \text{ n odd}, \\ \text{Tr } \gamma^5 \not{\phi}_1 \not{\phi}_2 \cdots \not{\phi}_n &= 0, \text{ n odd}, \\ \text{Tr } \gamma^5 &= 0, \\ \text{Tr } \gamma^5 \not{\phi}_1 \not{\phi}_2 &= 0, \\ \text{Tr } \not{\phi}_1 \not{\phi}_2 \not{\phi}_3 \not{\phi}_4 &= 4(a_1 \cdot a_2 a_3 \cdot a_4 - a_1 \cdot a_3 a_2 \cdot a_4 + a_1 \cdot a_4 a_2 \cdot a_3). \end{aligned} \quad (23)$$

We now relate O with $\not{D} = \not{\partial} + \not{B}$ by taking the square of this last quantity and letting it act on some twice differentiable function $f = f(x)$:

$$\begin{aligned} \not{D} \not{D} f &= \partial^2 f + \not{\partial}(\not{B} f) + \not{B} \not{\partial} f + B^2 f \\ &= \partial^2 f + \not{\partial}(\not{B} f) - \gamma^\mu \not{B} \partial_\mu f + 2B \cdot \partial + B^2 f \\ &= O f + (\not{\partial} \not{B}) f, \end{aligned}$$

or

$$\not{D} \not{D} = O + (\not{\partial} \not{B}). \quad (24)$$

We use this form of \mathcal{D}^2 in the traces on the right of (22). The first trace takes the form

$$\frac{1}{8}\text{Tr}^2 [O + (\partial\mathcal{B})] = 2(O^2 + O(\partial \cdot B) + (\partial \cdot B)O + (\partial \cdot B)^2), \quad (25)$$

and the second

$$\frac{1}{2}\text{Tr} [O + (\partial\mathcal{B})]^2 = 2(O^2 + O(\partial \cdot B) + (\partial \cdot B)O) + \frac{1}{2}\text{Tr} [(\partial\mathcal{B})(\partial\mathcal{B})]. \quad (26)$$

Recalling the formula (23) and using it for $a_1 = \partial$, $a_2 = B$, $a_3 = \partial$ and $a_4 = B$, one obtains

$$\frac{1}{8}\text{Tr}^2 [O + (\partial\mathcal{B})] - \frac{1}{2}\text{Tr} [O + (\partial\mathcal{B})]^2 = (\partial_{[\mu}B_{\nu]})(\partial^{[\mu}B^{\nu]}) \quad (27)$$

as we wished to demonstrate. There are two motivations for the including the first trace term in (22): it ensures that the differential operators be restrained, and it cancels some terms that look alien to high energy quantum field theories.

With the help of the Theorem, we can write the second term of (2), the QED Lagrangian, in the form

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = e^{-2} \left(\frac{1}{32}\text{Tr}^2\mathcal{D}^2 - \frac{1}{8}\text{Tr}\mathcal{D}^4 \right). \quad (28)$$

The point here is that we are using only the spinorial form of the covariant derivative, so that we can generalize the derivative.

4.2 Gauge field's kinetic energy using a spinorial covariant derivative: nonabelian case.

If we are dealing with a nonabelian YMT we have to revise the derivation of the previous subsection looking for places where the new noncommutativity of B_μ may make a difference. We begin with the same expression $\frac{1}{8}\text{Tr}^2\mathcal{D}^2 - \frac{1}{2}\text{Tr}\mathcal{D}^4$ and our interest is seeing if new terms have appeared in the right hand side of (27).

Notice that in (24) it is no longer true that $\mathcal{B}\mathcal{B} = B^2$, so that if we maintain the same definition for O the we must modify (26). On the other hand, (25) does not have to be changed, since it is still true that $\text{Tr } \mathcal{B}\mathcal{B} = 4B^2$. The new form of the second trace of the right hand side of (22) is

$$\frac{1}{2}\text{Tr}\mathcal{D}^4 = \frac{1}{2}\text{Tr} [O - B^2 + \mathcal{B}\mathcal{B} + (\partial\mathcal{B})]^2, \quad (29)$$

so that we obtain the

Theorem, nonabelian case. *Let $D_\mu = \partial_\mu + B_\mu$, where B_μ is a nonabelian vector field. Then:*

$$\begin{aligned} \frac{1}{8}\text{Tr}^2\mathcal{D}^2 - \frac{1}{2}\text{Tr}\mathcal{D}^4 &= 2(\partial \cdot B + B^2)^2 - \frac{1}{2}\text{Tr} [(\partial\mathcal{B}) + \mathcal{B}\mathcal{B}]^2 \\ &= ((\partial_{[\mu}B_{\nu]}) + [B_\mu, B_\nu])^2, \end{aligned} \quad (30)$$

where the traces are to be taken over the Dirac matrices.

The kinetic energy of a nonabelian gauge field, using the spinorial covariant derivative, can be written using the relation we have just derived. This part of the lagrangian of a nonabelian gauge field is given, with the help of the previous equation, by

$$\frac{1}{2g^2} \widetilde{\text{Tr}} (\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu])^2 = \frac{1}{2g^2} \widetilde{\text{Tr}} \left(\frac{1}{8} \text{Tr}^2 \not{D}^2 - \frac{1}{2} \text{Tr} \not{D}^4 \right).$$

4.3 Construction of a generalized Yang-Mills theory.

The group-theoretical setup for a GYMT is the same as for a YMT. We begin with a non-abelian local Lie group. Again for concreteness we choose the group to be $SU(N)$. Just as we did when constructing the YMT we put the gauge fields in the adjoint, so there must be N of them. We put the fermions in the fundamental \mathbf{N} , and use $\mathbf{N} \times \mathbf{N}^* = \mathbf{Adj} + \mathbf{1}$ to construct the invariant fermion energy term $\bar{\psi} i \not{D} \psi$.

To construct the GYMT we generalize the transformation of the gauge field as follows: to every generator in the Lie group we choose one gauge field that can be either vector or scalar, so that if we choose N_V generators to be associated with an equal number of vector gauge fields and N_S to be associated with an equal number of scalar gauge fields then $N_V + N_S = N$. If we choose a certain Lie group G to base the GYMT on, then the maximal subgroups of G are the possible different GYMTs for that choice of group. Both the form of the covariant derivative as well as the chiral structure of the theory are determined by this choice.

Let say that we have decided upon a particular maximal subgroup G_M , so that

$$G \supset G_M.$$

The way we construct the GYMT, if we restrict appropriately restrict the transformation group, we will have a normal YMT with G_M as gauge group. The constructions proceeds as follows. We define the generalized covariant derivative D by taking each one of the generators and multiplying it by one of its associated gauge fields and summing them together. The result is

$$D \equiv \not{\partial} + \not{A} + \Phi \tag{31}$$

where

$$\begin{aligned} \not{A} &= \gamma^\mu A_\mu = ig\gamma^\mu A_\mu^a T^a, & a = 1, \dots, N_V, \\ \Phi &= \gamma^5 \varphi = -g\gamma^5 \varphi^b T^b, & b = N_V + 1, \dots, N, \end{aligned} \tag{32}$$

and the generators T^a , $a = 1, \dots, N_V$, are the generators of the maximal subgroups. The scalar fields φ^b are associated only to those generators T^b that are generators of the original group G , but not of the maximal subgroup G_M . (Observe the difference between A_μ and A_μ^a , and between φ and φ^b .)

We now define the transformation for the gauge fields to be

$$\not{A} + \Phi \rightarrow U(\not{A} + \Phi)U^{-1} - (\not{\partial} U)U^{-1}, \tag{33}$$

from which one can conclude that

$$D \rightarrow UDU^{-1} \quad (34)$$

To construct the lagrangian we require that it should:

- contain only fermion fields and covariant derivatives;
- possess both Lorentz and gauge invariance;
- have units of energy to the fourth power.

There are not many lagrangians compatible with these requirements. There is a little freedom left in that one can still choose the irrep for the fermions to lie in. A generic such lagrangian is

$$\mathcal{L}_{GYM} = \bar{\psi}iD\psi + \frac{1}{2g^2}\widetilde{\text{Tr}}\left(\frac{1}{8}\text{Tr}^2D^2 - \frac{1}{2}\text{Tr}D^4\right), \quad (35)$$

where the trace with the tilde is over the Lie group matrices and the one without it is over matrices of the spinorial representation of the Lorentz group. The additional factor of 1/2 that the traces of (35) have with respect to (22) comes from the normalization given by (12), the usual one in non-abelian YMTs.

The expansion of this lagrangian into component fields results in expressions that are traditional in Yang-Mills theories. One has to substitute (31) in (35), and work out the algebra. It is convenient to first get the intermediate result

$$\begin{aligned} \frac{1}{16}\text{Tr}^2D^2 - \frac{1}{4}\text{Tr}D^4 &= ((\partial \cdot A) + A^2)^2 - \text{Tr}((\not{\partial}A) + A\not{A})^2 \\ &\quad - \frac{1}{4}\text{Tr}((\not{\partial}\Phi) + \{A, \Phi\})^2, \end{aligned} \quad (36)$$

where the curly brackets denote an anticommutator. Notice in this expression that the differentiation operators are restrained, and that the γ^5 present in the scalar boson Φ has the dual essential function of ensuring that the partials become restrained and the anticommutators commutators.

To finish the calculation we substitute (32) in (36) and in the fermionic term of (35) to obtain

$$\begin{aligned} \mathcal{L}_{GYM} &= \bar{\psi}(i\not{\partial} + A)\psi - g\bar{\psi}i\gamma^5\varphi^bT^b\psi + \frac{1}{2g^2}\widetilde{\text{Tr}}\left((\partial_{[\mu}A_{\nu]} + [A_{\mu}, A_{\nu}])^2\right) \\ &\quad + \frac{1}{g^2}\widetilde{\text{Tr}}((\partial_{\mu}\varphi + i[A_{\mu}, \varphi])^2). \end{aligned} \quad (37)$$

These are familiar structures: the first term on the right looks like the usual matter term of a gauge theory, the second like a Yukawa term, the third like the kinetic energy of vector bosons in a Yang-Mills theory and the fourth like the gauge-invariant kinetic

energy of scalar bosons in the non-abelian adjoint representation. If we restrict the transformation group G to the maximal subgroup G_M then those terms would not just “look” like a YMT, they would be a YMT. It is also interesting to observe that, if in the last term we set the vector bosons equal to zero, then this term simply becomes $\sum_{b,\mu} \frac{1}{2} \partial_\mu \varphi^b \partial^\mu \varphi^b$, the kinetic energy of the scalar bosons. We have constructed a generic non-abelian gauge theory with gauge fields that can be either scalar or vector.

4.4 Chiral structure.

We have called the expanded lagrangian “generic”, because it is the most common form of a GYMT lagrangian; nevertheless, different choices of irrep for the fermions will make its form vary slightly. Furthermore, we have not gone into the chiral structure detail. Let us give an schematic example. Suppose the maximal subgroup G_M is composed of two subgroups. The spinor ψ is then going to have some elements with right and some with left chiralities, depending on which one of the subgroups of the maximal subgroup G_M is acting upon the element. If, for the example, the connection looks like this:

$$\begin{pmatrix} \text{vector fields} & \text{scalar fields} \\ \text{scalar fields} & \text{vector fields} \end{pmatrix},$$

then the fermion interaction term in the lagrangian term will look like this:

$$(\bar{\psi}_L \quad \bar{\psi}_R) \begin{pmatrix} \mathcal{B} & \phi \\ \phi^\dagger & \mathcal{C} \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} i\mathcal{B}\psi_R + i\phi\psi_L \\ -i\phi^\dagger\psi_R + i\mathcal{C}\psi_L \end{pmatrix}. \quad (38)$$

In this expression $\bar{\psi}_L \equiv \overline{\psi_R}$ and $\bar{\psi}_R \equiv \overline{\psi_L}$, and we have written \mathcal{B} and \mathcal{C} to generically represent the vector gauge fields of the two subgroups that make up G_M , and ϕ to generically represent the scalar gauge fields. Notice that in the expansion of this matrix product the terms $\bar{\psi}_L \mathcal{B} \psi_R$ and $\bar{\psi}_R \mathcal{C} \psi_L$ have their respective spinor fields with opposite chirality, and that the terms $\bar{\psi}_R \phi \psi_R$ and $\bar{\psi}_L \phi \psi_L$ have their respective spinor fields with the same chiralities, as is usual in Yukawa terms.

Terms like $\bar{\psi}_L \mathcal{B} \psi_R$ reflect the usual meaning of connection: ψ_R suffers the effect of the connection and then it is multiplied the conjugate of its original self. On the other hand, a term like $\bar{\psi}_L \phi \psi_L$ shows that the new part of the connection is more like the space inversion discrete transformation in its effect than the usual connection. It does not involve a vector index to relate to a path, and it changes the chirality of the spinor, relating the left and right hand degrees of freedom, like the mentioned transformation.

5 A generalized Yang-Mills theory for Glashow-Weinberg-Salam: $U(3)$.

$SU(3)$ contains $U_Y(1) \times SU_I(2)$ as a maximal subgroup, so one wonders if it could be a unification group for the GWSM. Its diagonal generators correctly assign the

hypercharge and isospin quantum numbers to the electron and neutrino, but give the wrong hypercharge value to the Higgs bosons because of the sign of the last component of the hypercharge generator. Even worse, it does not make phenomenological sense at all to try to unify the electroweak model using $SU(3)$, because of the proliferation of physically nonexistent vector bosons. Yet some work has been done on $SU(3)$ as a unification group⁴, but our interest here is more along the lines of Y. Ne'eman and D.B. Fairlie,^{5,6} that considered using the graded group⁷ $SU(2/1)$, putting in the adjoint both the vector and the Higgs bosons of the GWSM. This way there are no extra vector bosons arriving from the unification; instead, the Higgs bosons nicely take those places.

The use of graded groups has, at least, two very serious problems. First, it gives anticommuting properties to the Higgs bosons. Second, due to the definition of the supertrace, at least one of the vector boson kinetic energy terms has the wrong sign.

The simpler choice of the non-graded group $SU(3)$ was forgone on the basis that it resulted in an incorrect prediction of the value of the Higgs boson's hypercharge. However, it was later pointed out that $U(3)$ has a non-standard representation that does give the correct hypercharge to the Higgs.⁸ The additional vector boson using $U(3)$ brings in, turns out to automatically decouple from the rest of the model and, interestingly enough, all the terms of the GWSM, except the Higgs boson potential $V(\varphi)$, come out correctly from just two terms and two irreps, one for bosons and one for fermions. So the door is opened to use scalar bosons as part of the covariant derivative.

5.1 Why $SU(3)$ does not work.

The GWSM is the product of two groups, each one with its own coupling constant. Thus $U_Y(1)$ has g' and $SU_I(2)$ has g . According to the model, they must be related by

$$g' = g \tan \theta_W,$$

where θ_W is Weinberg's angle. Its experimental value is $\theta_W \approx 29^\circ$. If instead we take $\theta_W = 30^\circ$ exactly, then the two coupling constants must obey

$$g' = g/\sqrt{3}. \quad (39)$$

The generator for $SU(3)$ that we want to associate with hypercharge is

$$T^8 = \frac{1}{2} \lambda^8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2), \quad (40)$$

and with isospin is

$$T^3 = \frac{1}{2} \lambda^3 = \frac{1}{2} \text{diag}(1, -1, 0).$$

These generators correspond to the vector bosons \mathcal{B} and \mathcal{A}^3 , that in the GWSM would belong to $U_Y(1)$ and $SU_I(2)$, respectively. The electric charge of a particle should be

given in terms of these two generators by the relation

$$Q = T_3 + \frac{1}{2}Y. \quad (41)$$

Here we have introduced the hypercharge operator Y . We have followed tradition and placed a factor of $\frac{1}{2}$ multiplying the hypercharge. Associating the boson fields $A^1, A^2, \varphi^4, \varphi^5, \varphi^6$ and φ^7 to the generators T^1, T^2, T^4, T^5, T^6 and T^7 , respectively, and defining $A_- \equiv A^1 - iA^2$, $\Phi_1 \equiv \Phi^4 - i\Phi^5$, $\Phi_2 \equiv \Phi^6 - i\Phi^7$, the covariant derivative of the group $SU(3)$ then looks like this:

$$D_{SU(3)} = i\partial + \frac{1}{2} \begin{pmatrix} A_3 + B/\sqrt{3} & A_- & \Phi_1 \\ A_+ & -A_3 + B/\sqrt{3} & \Phi_2 \\ \Phi_1^\dagger & \Phi_2^\dagger & -2B/\sqrt{3} \end{pmatrix}, \quad (42)$$

where the A and B fields both contain, as shown in (32), the same coupling constant. Notice, however, that since the B field belongs in the maximal subgroup $U(1)$, and this group is going to be identified with $U_Y(1)$, it is possible to include the factor of $\frac{1}{\sqrt{3}}$ in a redefinition of the coupling constant of the $U(1)$, precisely with form (39). Thus the T^8 generator is multiplied by $2\sqrt{3}$ leaving the appropriate operator for the hypercharge quantum number:

$$Y' = 2\sqrt{3}T^8 = \text{diag}(1, 1, -2). \quad (43)$$

As we shall very soon see, not only T^8 , but the other diagonal generators that show in the GYMT for the GWSM and have a nonzero $(3, 3)$ component fortunately always have a normalization coefficient of $1/\sqrt{3}$ which will allow us to use g' instead of g , and naturally mimic the two coupling constant structure of the GWSM.

If we go ahead and make the identification of the Y' operator we have constructed with the hypercharge operator, and of the $SU(3)$ T^3 generator with the isospin operator, then the electric charge operator is given by

$$Q' = \text{diag}(1, 0, -1), \quad (44)$$

according to (41). The GWSM fermions must then go in the chiral triplet

$$\psi_{SU(3)} = \begin{pmatrix} e_R^c \\ \nu_R^c \\ e_R \end{pmatrix}, \quad (45)$$

since this way they obtain the correct charge eigenvalues from (44).

Now let us find the hypercharge of the Higgs, which is given by the coefficients of the field itself after commutation with the hypercharge generator.

$$[Y', \Phi] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \Phi_1 \\ 0 & 0 & \Phi_2 \\ \Phi_1^\dagger & \Phi_2^\dagger & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 3\Phi_1 \\ 0 & 0 & 3\Phi_2 \\ -3\Phi_1^\dagger & -3\Phi_2^\dagger & 0 \end{pmatrix} \quad (46)$$

So we get the value 3 and not 1, that we know from the GWSM is the correct value. We conclude that it is not possible to get the GWSM from $SU(2/1)$.

Alternatively, it is possible to derive this result directly from Dynkin methods.⁸ Thus for $SU(3) \supset U_Y(1) \times SU_I(2)$ one has that $8 = 1_0 + 2_3 + \bar{2}_{-3} + 3_0$, where, for example, in 2_3 the 2 is the isospin multiplet and the subindex 3 the hypercharge. This means that the $SU(3)$ theory has four vector bosons (the three A^a and the B) and four scalar bosons (the two complex Higgs), and that the hypercharge of the Higgs is 3, like we calculated above.

5.2 Why $SU(2/1)$ does not work, either.

So it becomes clear why graded algebras were invoked. The $SU(2/1)$ graded algebra has eight generators, just like $SU(3)$, and all of them are precisely the same as this Lie group's except for

$$T^0 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, 2), \quad (47)$$

that differs from T^8 in the sign of the two. The point is that with this generator one then defines a different hypercharge operator

$$Y = 2\sqrt{3}T^0 = \text{diag}(1, 1, 2) \quad (48)$$

that results in a the correct hypercharge assignment for the Higgs:

$$[Y, \Phi] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \Phi_1 \\ 0 & 0 & \Phi_2 \\ -\Phi_1^\dagger & -\Phi_2^\dagger & 0 \end{pmatrix} \right]. \quad (49)$$

In spite of this promising beginning, it turns out that there are two fundamental problems about using graded groups. They are:

- the trace of their tensors of $SU(2/1)$ is not group-invariant. Only the supertrace is, but supertraces are not positive definite (they multiply by -1 some of the diagonal elements), and thus at least one of the kinetic energies of the vector bosons is going to have the wrong sign;
- the parameters that multiply the T^4, T^5, T^6 and T^7 generators of $SU(2/1)$ have to be Grassmann numbers, so that the Higgs fields have to be anticommuting among themselves, in total disagreement with the fundamentals of quantum field theory, that require scalar bosons to be commuting.

These problems are very serious, and with time interest in the model waned.⁹

5.3 The difference $U(3)$ makes.

While it is not possible to construct a GYMT for the GWSM using $SU(3)$, it is possible to do it using $U(3)$. The generators of this group are the same as those of $SU(3)$, but with an extra one that has to have a nonvanishing trace. To see the reason for this let $U = \exp(i\omega^a T^a)$, and notice that $\det U = \exp(i\omega^a \text{Tr } T^a)$. Since for $U(3)$ one has that, in general, $\det U \neq +1$, then at least one of the T^a has to be traceless. Take this operator to be

$$T^9 = \frac{1}{\sqrt{6}} \text{diag}(1, 1, 1). \quad (50)$$

With the two generators T^9 and T^8 it is possible to institute a new representation of $U(3)$. This new representation would contain the generators T^1, \dots, T^7 of $SU(3)$, the $SU(2/1)$ generator T^0 seen in (47), and a new generator

$$T^{10} = \frac{1}{\sqrt{6}} \text{diag}(1, 1, -1). \quad (51)$$

The generators T^0 and T^{10} can be rewritten as a linear combination of the two original $U(3)$ generators T^8 and T^9 , as follows:

$$\begin{aligned} T^{10} &= \frac{2\sqrt{2}}{3} T^8 + \frac{1}{3} T^9, \\ T^0 &= -\frac{1}{3} T^8 + \frac{2\sqrt{2}}{3} T^9. \end{aligned}$$

Notice that T^{10} and T^0 are orthogonal under (12).

The nine generators must correspond to nine boson fields. The boson Υ we are to associate with the new generator T^{10} should be scalar, since this generator is not part of the maximal subgroup. What we are seeing in our example is that the original group has a maximal subgroup $G_M = U(1) \times U(1) \times SU(2)$ in the sense that it is both a subgroup and there is no subgroup larger than it, that is still properly contained in G . And what we are forced to do to get the quantum numbers right is to take a linear combination of those two $U(1)$'s, which is equivalent to taking the original representation and using fields that are a mixture of scalar and vector bosons, something like $A_\mu \gamma^\mu + \phi \gamma^5$. It is interesting that a similar thing happens again when we attempt grand unification.

From a phenomenological point of view it is clear we must take the new boson Υ to be a scalar, since this way it is more likely it will decouple from the other scalar bosons and the fermions. A vector boson will inevitably couple with the other vector bosons and show up phenomenologically. This decoupling of the scalar bosons from each other, and of the diagonal scalar bosons from the fermions, is a remarkable property of GYMTs, and we shall study it next for the case of the GWSM.

5.4 Emergence of the Glashow-Weinberg-Salam model from $U(3)$.

Let us briefly review what has happened, but now using branching rules for maximal subalgebras.¹⁰ If one takes $SU(3)$ as GYMT then $SU(3) \supset U_Y(1) \times SU_I(2)$. The fermions are then placed in the fundamental **3** that has a branching rule **3** = **1₋₂**+**2₁** (the hypercharge appears as subindex), that is, we end up with an isospin singlet with $Y = -2$ and an isospin doublet with $Y = 1$. This implies the fermion multiplet has to be chiral and of the form (45). The problem in this case lies elsewhere. The bosons are in the adjoint **8** that has the branching rule **8** = **1₀**+**2₃**+**2₋₃**+**3₀**, so that we end up with a vector boson singlet, a vector boson triplet, and a complex scalar doublet, the usual boson spectrum in the GWSM. However, as we mentioned, this picture is unacceptable because the hypercharge of the Higgs doublet is $Y = 3$, as can be seen from the branching rule. So, instead, we take the group $U(3)$ with the maximal subgroup $U(3) \supset U_{Z'}(1) \times U_{Y'}(1) \times SU_I(2) \rightarrow U_Z(1) \times U_Y(1) \times SU_I(2)$, where we have called Z' and Y' the quantum numbers associated with the gauge bosons of the two $U(1)$'s, and Z and Y are generators that are linear combinations of those numbers and give a particle spectrum with the same quantum numbers of the GWSM. The electric charge operator that obtains through the use of the hypercharge Y of equation (48) is given by

$$Q = \text{diag}(1, 0, 1),$$

that implies that the fermion multiplet is the nonchiral

$$\psi_{U(3)} = \begin{pmatrix} e_R^c \\ \nu_R^c \\ e_L^c \end{pmatrix}. \quad (52)$$

This choice of hypercharge assigns to all the particles that come out of the $U(3)$ GYMT the correct quantum numbers of the GWSM.

Let us see how it is that the extra scalar boson Υ decouples in this case. From an inspection of (37) we see scalar bosons in GYMT do not interact among themselves. Those terms seem to be included in (35), but then disappear for algebraic reasons and are absent in (37). Scalar bosons only appear in the expanded generalized lagrangian in the Yukawa term and in the gauge invariant kinetic energy term. The interactive part of this term is proportional to

$$\widetilde{\text{Tr}}((A_\mu, \Phi)^2). \quad (53)$$

The positioning of the fields in the adjoint representation is illustrated in the matrix:

$$\begin{pmatrix} \text{Vector fields and } \Upsilon & \text{Higgs fields } \varphi \\ \text{Higgs fields } \varphi & \text{Vector fields and } \Upsilon \end{pmatrix}$$

The commutator $[A_\mu, \Phi]$ is certainly not zero when Φ is one of the usual GWSM Higgs bosons φ because these bosons occupy places off the block diagonal, but when

Φ is the scalar boson Υ , that only has components along the diagonal and that is proportional to the identity matrix in each of those boxes, then the commutator (53) has to be zero.

The decoupling of the Υ from the fermions is due to similar reasons. The scalar bosons only couple to the fermions through the Yukawa terms, which are proportional to $\bar{\psi}\Phi\psi$. If Φ is one of the Higgs bosons φ then it occupies a block that is off the matrix' diagonal and so it stands between fermion spinors of the same chirality. But the Υ connects spinors of the opposite chirality and thus this term is zero.

5.5 The lagrangian of the Glashow-Weinberg-Salam model.

The covariant derivative of the $U(3)$ GYMT we have been developing has the form

$$D_{SU(3)} = i\partial + \frac{1}{2} \begin{pmatrix} igA^a\sigma^a + g'iB & -\sqrt{2}\gamma^5\hat{\varphi} \\ -\sqrt{2}\gamma^5\hat{\varphi}^\dagger & 2ig'B \end{pmatrix},$$

where $(\sigma^a) = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices, and

$$\hat{\varphi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi^4 - i\varphi^5 \\ \varphi^6 - i\varphi^7 \end{pmatrix}.$$

The GWSM lagrangian follows from the application of this covariant derivative to lagrangian (37):

$$\begin{aligned} \mathcal{L}_{GWSM} = & \overline{\theta_R^c}(i\partial - \frac{1}{2}gA^a\sigma^a - \frac{1}{2}g'B)\theta_R^c + \overline{e_L^c}(i\partial - g'B)e_L^c \\ & + i\frac{\sqrt{2}}{2}g\overline{e_L^c}\hat{\varphi}^\dagger\theta_R^c - i\frac{\sqrt{2}}{2}g\overline{\theta_R^c}\hat{\varphi}e_L^c - \frac{1}{4}\mathbf{A}_{\mu\nu}\cdot\mathbf{A}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ & + \left|(\partial_\mu + \frac{1}{2}igA_\mu^a\sigma^a - \frac{1}{2}ig'B_\mu)\hat{\varphi}\right|^2 + \frac{1}{2}(\partial_\mu\Upsilon)^2, \end{aligned}$$

where we have employed the fermion multiplet $\psi = (52)$ and we have defined the Higgs doublet $\theta_R^c = \begin{pmatrix} e_R^c \\ \nu_R^c \end{pmatrix}$. It is the usual GWSM lagrangian's, except that the Higgs potential $V(\hat{\varphi})$ is missing. The extra scalar boson does not interact at all.

6 Grand unification using generalized Yang-Mills theories.

In this section we shall describe how to build a GUT using GYMTs, and construct one such example based on the GYMT $SU(6)$, that is not that much larger group than $SU(5)$. It would seem at first sight that the logical choice for a generalized GUT would be to use the maximal subgroup $SU(6) \supset SU(3) \times SU(3) \times U(1)$, because we know that a $SU_C(3)$ is needed to model quantum chromodynamics.¹¹ However, this

choice is a step in the wrong direction and makes the algebra very messy. The correct way is to take $SU(6) \supset SU(5) \times U(1)$, which basically results in the usual $SU(5)$ GUT.¹² We shall not cover other GYMT grand unification schemes here.

A word of clarification: in general there is a difference in the “unification” brought about in a GUT or a GYMT. In a GUT two or more forces are unified choosing a large Lie group that has as products in one of its maximal subgroups the Lie groups that represent the forces being unified. Usually the breaking of the symmetry into the original forces is achieved through a Higgs field multiplet with one component that has a nonzero VEV. In the case of a GYMT the “unification” that is achieved includes the usual one we have just described *plus* an additional, rather technical one, that consists in having only two terms and two irreps in the lagrangian. Compare this with, for example, the $SU(5)$ GUT. It has two irreps for the fermions, one for the vector bosons, and two for the Higgs fields, for a total of five irreps. The terms in the lagrangian are: the kinetic energy of the vector bosons, the kinetic energy of the two types of Higgs, two types of Yukawa terms, the kinetic energies of the fermions, and potentials for the Higgs fields, for a total of nine types of terms. On the other hand, there is one problem with GYMTs: there are no potentials for the Higgs fields.

6.1 The Yang-Mills $SU(5)$ GUT.

We are going to briefly review the $SU(5)$ GUT, particularly aspects that are of relevance to our topic. It is a Yang-Mills theory based on $SU(5)$ with the vector bosons placed in the adjoint **24**, and the fermions in two irreps which are the conjugate fundamental **5** and an antisymmetric **10**. The Higgs bosons that break the symmetry in the GWSM occupy another fundamental **5**, and the Higgs bosons that give very high masses to the unseen vector bosons occupy another adjoint **24**.

The breaking of the $SU(5)$ symmetry is into the $SU_C(3) \times SU_I(2) \times U_Y(1)$ maximal subgroup. The branching rule for the vector bosons is

$$\mathbf{24} = (\mathbf{1}, \mathbf{1})_0 + (\mathbf{3}, \mathbf{1})_0 + (\mathbf{2}, \mathbf{3})_{-5} + (\mathbf{2}, \bar{\mathbf{3}})_5 + (\mathbf{1}, \mathbf{8})_0, \quad (54)$$

where we have placed the hypercharge as a subindex. The 12 vector bosons with hypercharge ± 5 are not seen phenomenologically, so they are presumed to couple with the Higgs of the **24**. The branching rule for the Higgs bosons is the same as for the vector bosons, since they occupy the same irrep. Of these **24** Higgs only one could have a nonzero VEV and that is the $(\mathbf{1}, \mathbf{1})_0$, because it has zero electric and color charges. Otherwise the vacuum would also have these quantum numbers, and it does not. The generator associated with this field is thus proportional to the matrix

$$\text{diag}(2, 2, 2, -3, -3),$$

according to the rule of obtaining the quantum numbers of fields in a matrix by taking its commutation with the diagonal generators. We shall see how this same Higgs shows up quite unexpectedly in the $SU(6)$ GYMT.

The group $SU(5)$ has a symmetric **15** irrep. Is it possible to place the fermions in this single irrep instead of using the two irreps **5** and **10**? The answer, within a usual Yang-Mills theory, is no, the reason being that the quantum numbers of the particles would force the irrep to be nonchiral, and thus, unacceptable. However, in a GYMT, not only is it *possible* to place all the fermions in the **15**, it is *required* by the theory as we shall immediately see.

6.2 The $SU(6) \supset SU(5) \times U(1)$ generalized grand unified theory.

The low-dimensional irreps of $SU(6)$ are the fundamental **5**, the antisymmetric **15**, the symmetric **21** and the adjoint **35**. Since we have **15** fermions the only choice we have is the **15**, but this implies a nonchiral irrep. In a normal YMT this would have been a headache, since we would need the usual **5*** and **10** structure. But the **15** fermion irrep is precisely what we need in a GYMT, since it implies a nonchiral irrep the way we need it precisely.

6.3 The boson term.

We use the maximal subgroup $SU(6) \supset SU(5) \times U(1)$. The branching rule for the **35** is

$$\mathbf{35} = \mathbf{1}_0 + \mathbf{5}_6 + \bar{\mathbf{5}}_{-6} + \mathbf{24}_0,$$

where we have written the quantum number associated with the $U(1)$ as a subindex. The $\mathbf{24}_0$ contains the 24 vector bosons of the $SU(5)$ GUT, 12 of which are the vector bosons of the GWSM and quantum chromodynamics and the other 12 the X carriers of the grand unification force. The two **5**'s are the Higgs bosons that later branch into the GWSM Higgs doublet; they are exactly the same irrep we just examined in the $SU(5)$ GUTS, but now branching from the $SU(6)$'s **35**. Therefore we call them the φ , again. Finally the $\mathbf{1}_0$ is the scalar boson we shall call Ω , which corresponds to the generator of a quantum number we shall call ultracharge. It has the form

$$T_6^U \propto \text{diag}(1, 1, 1, 1, 1, -5). \quad (55)$$

Here we have a big difference between Yang-Mills and generalized Yang-Mills grand unification: in the former case the heavy (with a mass of the order of the grand unification energy scale) Higgs are in a **24**, in the latter there is only one and in a singlet. The covariant derivative is then, schematically,

$$D_{SU(6)} = \partial + \begin{pmatrix} \mathbf{24} + \Omega & \varphi \\ \varphi^\dagger & -5\Omega \end{pmatrix}.$$

Notice that the color and electric charge generators, which all have a zero $(6, 6)$ component and are diagonal in the 5×5 block, commute with T^U so that Ω is completely neutral and can take a VEV. This is consistent with the fact it is a $SU(5)$ singlet.

6.4 The fermion term.

The fermions are in the irrep **15** that is an antisymmetrization of the Kronecker product of two fundamental irreps, that is, $\mathbf{15} = \mathbf{6} \times \mathbf{6}|_a$. Let us construct the invariant term that contains the fermions. It is going to be slightly different from the generic case we considered when we gave the construction of GYMTs. The fermion Hilbert space in this case is the 6×6 matrix ψ formed by contracting the fermion fields with the Clebsch-Gordan matrix relating $\mathbf{6} \times \mathbf{6}|_a$ with the **15**. This happens to be an antisymmetric 6×6 matrix that maintains its antisymmetry under the transformation $\psi \rightarrow U\psi U^T$, or, equivalently, $\bar{\psi} \rightarrow U^*\bar{\psi}U^{-1}$. The covariant derivative transforms as $D \rightarrow UDU^{-1}$, so the gauge-invariant term must then be

$$\widetilde{\text{Tr}}(\bar{\psi}iD\psi). \quad (56)$$

To prove the invariance of this term notice that under a group transformation,

$$\widetilde{\text{Tr}}(\bar{\psi}iD\psi) \rightarrow \widetilde{\text{Tr}}U^*\bar{\psi}U^{-1}UDU^{-1}U\psi U^T = \widetilde{\text{Tr}}(\bar{\psi}iD\psi),$$

since all the U 's cancel.

The fermion Hilbert space vector under the $SU(6) \supset SU(5) \times U(1)$ maximal subalgebra, looks like the following 6×6 antisymmetric matrix, with the **5**_R being a column vector with 5 components and the **10** a 5×5 antisymmetric matrix:

$$\psi = \begin{pmatrix} \mathbf{10}_L & \mathbf{5}_R \\ -\mathbf{5}_R^T & \mathbf{0} \end{pmatrix}. \quad (57)$$

Again this is a nonchiral multiplet, as it always seems to be the case when GYMTs are involved.

6.5 A revision of the $SU(6)$ model.

Till now we have followed the recipe for constructing a GYMT for $SU(6)$ and have found that the $SU(5)$ GUT seems to follows from it. Does it really? First of all, although both models have the same dynamics, the GYMT version has additional symmetries that can be useful as custodial symmetries and have important physical consequences. Second, there is a problem we shall now study and has to do with one difference between the usual GUT and the GYMT.

The Higgs bosons that supply the grand unification mass scale occupy a **24** in the $SU(5)$ GUT. Under the maximal subgroup $SU(5) \supset SU_C(3) \times SU_I(2) \times U_Y(1)$ the branching rule for this irrep is (54), where the hypercharge as a subindex. It is usually assumed in GUTs that these scalar bosons are not observed because their masses are of the order of their VEVs, a situation that would make them unobservable at present. The component that has the nonzero VEV is the $(1, 1)_0$ Higgs, that corresponds to the generator

$$T_6^Y \propto \text{diag}(2, 2, 2, -3, -3, 0), \quad (58)$$

because it has no color or electric charge. We have given this generator the name T_6^Y because it evidently fulfills the same function in $SU(6)$ as the hypercharge generator $T^Y \propto \text{diag}(2, 2, 2, 3, 3)$ did in $SU(5)$, aside from the additional dimension. But this means that the hypercharge B_μ vector boson should also be assigned it. This is not quite our previous result, since we have the Ω assigned to the $\propto \text{diag}(1, 1, 1, 1, 1, -5)$ generator. This difficulty has a solution reminiscent of the way the original problem of finding a GYMT for GWSM was handled. In that case, as we have seen, the $U(3)$ group was used instead of $SU(3)$, and the correct model was obtained through a nonstandard representation of the group generators that would have been impossible with simply $SU(3)$. To solve our problem we have to resort to a similar maneuver: we must associate with the $SU(6)$ hypercharge generator T_6^Y a field Ξ that is a mixture of the B_μ hypercharge vector field and of the Ω Higgs. This field is given by

$$\Xi = \not{B} + \gamma^5 \Omega.$$

This change is enough to solve the problem since now the leptoquarks acquire the large grand unification mass, while at the same time there is still a vector boson that generates the hypercharge quantum number. We revert to the usual $SU(5)$ GUT.

Another interesting point to notice is that due to peculiar algebraic properties of GYMTs the two different types of Higgs bosons do not couple to each other, as can be appreciated from (37).

6.6 Serendipitous cancellations.

The coupling between fermions and vector bosons is given by the block structure of the matrices. This structure is dictated by the maximal subgroup that has been chosen and afterwards there is basically no more freedom of choice for us. Yet the results are very interesting. Another point that deserves attention is the way the formalism manages the correct coupling of fermions and Higgs bosons. The light Higgs bosons have to couple to the fermions in order to achieve mass generation, but the heavy Higgs cannot since that would give the fermions a large mass. The interaction term is given by (56) with (57) replaced in it. A careful analysis of the 178 interaction terms that result from taking the trace of the product of matrices shows that all the terms that should vanish do, and the ones that do not, have the correct couplings. It is remarkable result, and one that involves many serendipitous cancellations.

7 Geometrical considerations.

The curvature of a YMT can be found as we have already done for a Riemann manifold. We take a small closed curve C as we did in Section 3 and calculate the parallel displacement of a vector caused by a connection A_μ (instead of the Christoffel symbol). The result is the same as the one we obtained in that section; that is, that the curvature is the commutator of the covariant derivative. In other words, the curvature

of a YMT is given by $R_{a\mu\nu}^b = -[D_\mu, D_\nu]_{ab}$ or, omitting the matrix subindices and using the usual symbol $(F_{\mu\nu})_{ab} = -R_{a\mu\nu}^b$, as

$$F_{\mu\nu} = [D_\mu, D_\nu],$$

where D_μ is given by (9). In the YMT lagrangian the kinetic energy of the gauge bosons is proportional to the gauge trace of the square of the curvature

$$\mathcal{L}_{YMT} = \frac{1}{2g^2} \widetilde{\text{Tr}} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2g^2} \widetilde{\text{Tr}} ((\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu])^2).$$

Our interest is to generalize this mathematics to a GYMT. We already know from (30) how to write the Yang-Mills kinetic energy using spinors. The natural generalization would be then to say that the kinetic energy related to the curvature square of a GYMT is given by

$$\mathcal{L}_{GYMT} = \frac{1}{2g^2} \widetilde{\text{Tr}} \left(\frac{1}{8} \text{Tr}^2 D^2 - \frac{1}{2} \text{Tr} D^4 \right).$$

So the question is, what is the curvature in a GYMT? What is the equivalent of a Yang-Mills' $F_{\mu\nu}$ for a such a theory?

7.1 The generalized curvature.

The quantity that we want to call the generalized curvature should have the properties we have come to regard as indispensable. It has to be gauge and Lorentz invariant and it should not have unrestrained derivatives. Due to the scarceness of mathematical constructs in GYMTs, it can only be made of covariant derivatives. The best candidate for a generalized curvature is

$$F = \frac{1}{4} \text{Tr} D^2 - D^2. \quad (59)$$

(Following our previous convention, the trace is over the Dirac matrices.) We can expand this quantity using (31). The square of the covariant derivative is

$$D^2 = \partial^2 + \mathcal{A}^2 + \varphi^2 + (\not{\partial}\mathcal{A}) + 2A \cdot \partial + (\not{\partial}\Phi) + \{\mathcal{A}, \Phi\} \quad (60)$$

and

$$\frac{1}{4} \text{Tr} D^2 = \partial^2 + A^2 + \varphi^2 + \partial \cdot A + 2A \cdot \partial. \quad (61)$$

It is clear that for a nonabelian vector gauge field A^2 and \mathcal{A}^2 are not equal, nor is $\{\mathcal{A}, \Phi\}$ zero. It is straightforward to calculate that

$$F = \partial \cdot A - (\not{\partial}\mathcal{A}) + A \cdot A - \mathcal{A}\mathcal{A} - (\not{\partial}\Phi) - \{\mathcal{A}, \Phi\}. \quad (62)$$

Thus F is composed of restrained differential operators. Since D transforms as:

$$D \rightarrow UDU^{-1} \text{ or } D \rightarrow SDS^{-1},$$

under the gauge or Lorentz group, respectively, it is obvious that the curvature would also transform as

$$F \rightarrow UFU^{-1} \text{ or } F \rightarrow SFS^{-1}.$$

The Lorentz trace of the square of the curvature is easy to calculate:

$$\begin{aligned} \mathrm{Tr} F^2 &= \mathrm{Tr} \left(\left(\frac{1}{4} \mathrm{Tr} D^2 \right)^2 - \frac{1}{4} (\mathrm{Tr} D^2) D^2 - \frac{1}{4} D^2 (\mathrm{Tr} D^2) + D^4 \right) \\ &= \mathrm{Tr} D^4 - \frac{1}{4} (\mathrm{Tr} D^2)^2. \end{aligned}$$

Thus the kinetic energy of a GYMT, written in terms of the curvature, is

$$\mathcal{L}_{GYMT} = \frac{1}{2g^2} \widetilde{\mathrm{Tr}} \left(\frac{1}{8} (\mathrm{Tr} D^2)^2 - \frac{1}{2} \mathrm{Tr} D^4 \right) = -\frac{1}{4g^2} \widetilde{\mathrm{Tr}} \mathrm{Tr} (F^2).$$

So with GYMTs it goes as with YMTs: the kinetic energy term of the lagrangian can be written in terms of the square of the curvature.

7.2 The meaning of the generalized curvature.

We proceed to calculate the parallel transport around a small closed path. For a YMT parallel transport with a connection A_μ around a small parallelogram made up of two vectors x^μ and y^μ can be calculated in a way similar to our calculation of Subsection 3.3 in a Riemann manifold. Here we are dealing with flat spacetime, but the connection A_μ still does rotate vectors when they move. The area two-form is $\Delta S^{\mu\nu} \equiv \frac{1}{2} x^{[\mu} y^{\nu]}$, as before. The product of two vectors x^μ and y^μ is defined to be

$$x \cdot y \equiv \frac{1}{4} \mathrm{Tr} \not{x} \not{y} = x^\mu \eta_{\mu\nu} y^\nu,$$

where $\eta_{\mu\nu}$ is the metric of flat spacetime.

The generalized product of vectors times curvature in our formalism, following the pattern of substituting inner products by traces over contracted vectors, would logically be

$$S \cdot F \equiv \frac{1}{4} \mathrm{Tr} F \not{x} \not{y}.$$

Substituting for F from (62), and with the help of (23) and the trace theorems of (23) we can derive

$$\begin{aligned} S \cdot F &= \frac{1}{4} \mathrm{Tr} (\partial \cdot A - (\not{\partial} \not{A}) + A \cdot A - \not{A} \not{A} - (\not{\partial} \Phi) - \{\not{A}, \Phi\}) \not{x} \not{y} \\ &= (x \cdot (\partial A) \cdot y - y \cdot (\partial A) \cdot x + x \cdot A A \cdot y - y \cdot A A \cdot x) \\ &= x^\mu y^\nu (\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]}) = S^{\mu\nu} F_{\mu\nu}, \end{aligned}$$

which is the usual Yang-Mills result. This means that the straightforward guess gives the correct generalization, or, at least, is a step in the right direction.

7.3 The curvature in five dimensions.

The curvature we obtained for a GYMT is thus just the same as if we were dealing with a usual YMT. This is not a satisfying situation; information has been lost because of the lone γ^5 in the traces that makes many of them zero. The scalar fields turns out to be a useless bystander.

The origin of the problem lies in the lack of symmetry between the covariant derivative, that has five types of gauge fields (A_μ and φ), while only four types of derivatives (∂_μ). This asymmetry carries over to the vector \not{x} that has no γ^5 component. This situation results in many cancellations. From a mathematical point of view the esthetically pleasing thing to do is to add a new coordinate, which we call x^5 , and is to be interpreted as some sort of new dimension. The parallelogram around which to perform the parallel transport is then in a five-dimensional space, and the spinor form of the coordinates is:

$$\hat{x} \equiv \not{x} + ix^5\gamma^5 \quad \text{and} \quad \hat{y} \equiv \not{y} + iy^5\gamma^5. \quad (63)$$

It is important that

$$\frac{1}{4} \text{Tr } \hat{x}\hat{y} = x \cdot y - x^5y^5, \quad (64)$$

a result consistent with our ideas that adding a scalar multiplied by a γ^5 makes sense both physically and mathematically. Notice that the last term in (64) has a negative sign, so that its sign is spacelike, not timelike. In this respect it is pertinent that the term $x^5\gamma^5$ is not an observable, being antihermitian. Let us explain this comment by considering the electromagnetic interaction term

$$\mathcal{L}_{EM} = \bar{\psi} \not{A}_{EM} \psi.$$

The quantity \mathcal{L}_{EM} is real. To see this we take its hermitian conjugate, and through the following elementary sequence arrive at itself again: $\mathcal{L}_{EM}^* = \psi^\dagger \not{A}_{EM}^\dagger \gamma^0 \psi = \psi^\dagger \gamma^0 \not{A}_{EM} \psi = \mathcal{L}_{EM}$, since $\gamma^0 \gamma^\mu = \gamma^\mu \gamma^0$. If instead of \not{A} we had used \not{x} the result would have exactly the same. However, if we had simply used $x^5\gamma^5$, without the i , then $(\bar{\psi}x^5\gamma^5\psi)^* = (\bar{\psi}x^5\gamma^5\psi)^\dagger = -\bar{\psi}x^5\gamma^5\psi$, since $\gamma^{5\dagger} = \gamma^5$ and $\gamma^0\gamma^5 = -\gamma^5\gamma^0$. This is the reason that the last terms on the right in (63) require an i , with the implication that the fifth dimension is spacelike, not timelike.

When we perform parallel transport with the fifth dimension added the situation changes drastically. What we have now is a five-dimensional parallelogram, so that, calling S_5 the five-dimensional area element, the change in vector due to parallel transport around S_5 is:

$$S_5 \cdot F = \frac{1}{4} \text{Tr } F \hat{x} \hat{y}.$$

With respect to the previous calculation with the four dimensional parallelogram,

this one has an extra term that we can calculate:

$$\begin{aligned}
(S_5 - S_4) \cdot F &= \frac{1}{4} \text{Tr } F(\not{x}y^5\gamma^5 + x^5\gamma^5\not{y} + x^5y^5) \\
&= -\frac{1}{4} \text{Tr } ((\not{\partial}\Phi)\not{x}y^5\gamma^5 + (\not{\partial}\Phi)x^5\gamma^5\not{y} + \{\mathcal{A}, \Phi\}\not{x}y^5\gamma^5 + \{\mathcal{A}, \Phi\}x^5\gamma^5\not{y}) \\
&= \frac{1}{4} \text{Tr } ((\not{\partial}\varphi)\not{x}y^5 - (\not{\partial}\varphi)x^5\not{y} + [\mathcal{A}, \varphi]\not{x}y^5 - [\mathcal{A}, \varphi]x^5\not{y}) \\
&= x \cdot (\partial\varphi)y^5 - y \cdot (\partial\varphi)x^5 + x \cdot [A, \varphi]y^5 - y \cdot [A, \varphi]x^5 \\
&= (x^\mu y^5 - x^5 y^\mu)((\partial_\mu \varphi) + [A, \varphi]).
\end{aligned}$$

This result has to be added to the previous one of $x^\mu x^\nu F_{\mu\nu}$, and it is precisely This extra term employs the scalar field much in the same way as the vector fields the vector fields of a YMT.

8 Other implications of the new dimension.

The use of five coordinates in $\hat{x} \equiv \not{x} + x^5\gamma^5$ has an immediate implication we must carefully consider now. If there is another dimension there can be motion in that direction and so we must modify also the covariant derivative. We have been using $D = \not{\partial} + \mathcal{A} + \Phi$, but instead now we are going to use

$$\hat{D} = D + i\gamma^5\partial_5 = \not{\partial} + i\gamma^5\partial_5 + \mathcal{A} + \Phi, \quad (65)$$

where $\partial_5 = \partial/\partial x^5$. We have preferred to give the new coordinate the name x^5 and not x^4 , because it has to be associated with the γ^5 . This form of the covariant derivative is more symmetrical, since both the partials and the gauge fields benefit from a scalar term that is then multiplied by γ^5 . This new covariant derivative changes some of our previous results.

8.1 The new five-dimensional curvature.

The curvature itself is affected by the new form the covariant derivative has. Surprisingly, it differs by only one term from the previous four-dimensional GYMT curvature. All the other terms either cancel or vanish according to the formulae of (23). The new curvature \hat{F} is given by

$$\hat{F} = \frac{1}{4} \text{Tr } \hat{D}^2 - \hat{D}^2,$$

and

$$\hat{D}^2 = D^2 + i\gamma^5(\partial_5 \mathcal{A}) + i\{\varphi, \partial_5\} - \partial_5^2. \quad (66)$$

(Notice that the terms $\{\varphi, \partial_5\} + \partial_5^2$ are Lorentz scalars. The reader that has followed closely previous typical derivations may have observed that squared scalars of the covariant derivative usually cancel in GYMT's calculations, so that probably there

are not going to be terms like $\partial_5\varphi$ in the result.) Taking the trace of the previous equation we get

$$\frac{1}{4} \text{Tr } \hat{D}^2 = \frac{1}{4} \text{Tr } D^2 + i\{\varphi, \partial_5\} - \partial_5^2, \quad (67)$$

so that

$$\begin{aligned} \hat{F} &= F - i\gamma^5(\partial_5 A) \\ &= \partial \cdot A - (\partial A) + A \cdot A - A A - (\partial \Phi) - \{A, \Phi\} - i\gamma^5(\partial_5 A). \end{aligned} \quad (68)$$

We can now recalculate the parallel transport over a small five-dimensional parallelogram. The result come out to be:

$$\begin{aligned} S_5 \cdot \hat{F} &\equiv \frac{1}{4} \text{Tr } \hat{F} \hat{x} \hat{y} \\ &= \frac{1}{4} \text{Tr } F \hat{x} \hat{y} + \frac{1}{4} \text{Tr } i\gamma^5(\partial_5 A)(\not{y}^5 \gamma^5 + x^5 \gamma^5 \not{y}) \\ &= S_5 \cdot F - i(\partial_5 A_\mu)(x^\mu y^5 - x^5 y^\mu). \end{aligned}$$

Putting all the terms together, the parallel transport is

$$S_5 \cdot \hat{F} = S_{\mu\nu}(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]}) + S_{\mu 5}((\partial_\mu \varphi) - i(\partial_5 A_\mu) + [A, \varphi]) \quad (69)$$

where $S_{\mu 5} = (x^\mu y^5 - x^5 y^\mu)$. This looks very much like the curvature of a YMT in five dimensions, with the scalar fields φ taking the role of the fifth component of the gauge field A_μ . The difference is that in a real YMT in five dimensions each generator T^a is multiplied with all five $\mu = 0, 1, 2, 3, 5$ components of the gauge field A_μ^a , while in a GYMT a generator T^a is multiplied with either the first four components $\mu = 0, 1, 2, 3$ of the vector field or with an individual scalar field. In a five-dimensional YMT the number of generators T^a is the same as the number of five-dimensional vector fields, while in a GYMT roughly half the number of generators of the Lie group are associated with the four-dimensional vector fields A_μ^a , and half with the scalar fields φ^b .

The interest of result (69) is that it uses the extra scalar field. This is evidence in favor of the five-dimensional quantity \hat{x} . In the next subsections the theme is further examined.

8.2 A unexpected meaningful quantity.

If one looks at the way the results of the previous subsection were obtained, one realizes that GYMTs may possess an additional physical quantity that YMTs do not. For GYMTs it makes sense to calculate a different kind of “curvature” that involves only one dimension instead of two.

Consider the new quantity $\text{Tr } \hat{F} \hat{x}$, with only one coordinate segment. Using (68), (63) and (23) it is easy to verify that $\text{Tr } \hat{F} \hat{x} = 0$. However, it does make sense to calculate the quantity $\text{Tr } \hat{F} \hat{x} i\gamma^5$, that has an extra chirality matrix factor. If we were dealing with a YMT, then F would have only the terms $\partial \cdot A - (\partial A) + A \cdot A - A A$, and

$\text{Tr}F\cancel{x} = \text{Tr}F\cancel{x}i\gamma^5 = 0$. So within a YMT it does not make sense to define quantities of this kind, but within a GYMT it does. Let us calculate this new quantity:

$$\frac{1}{4}\text{Tr}\hat{F}i\hat{x}\gamma^5 = ix^\mu ((\partial_\mu\varphi) - i(\partial_5 A_\mu) + [A_\mu, \varphi]). \quad (70)$$

As it frequently happens in GYMTs, a formula that involves many terms ends up with relatively few. Notice this quantity is Lorentz- and gauge-invariant, and has no unrestrained coordinates. Every term on the right includes a structure that is present in GYMTs but is not in YMTs, so there can be no counterpart to this quantity in the latter theories. It looks like a local gauge-invariant dilation.

8.3 Changes in the lagrangian due to the fifth dimension.

The extra term in the covariant derivative makes absolutely no difference in the fermion sector. To verify this statement recall that this sector is composed of the single term

$$\mathcal{L}_F = \bar{\psi}i\hat{D}\psi = \bar{\psi}iD\psi - \bar{\psi}\gamma^5\partial_5\psi. \quad (71)$$

The first term on the right is the usual one in GYMTs, and the second one gives no contribution since

$$\bar{\psi}\gamma^5\partial_5\psi = \bar{\psi}_L\gamma^5\partial_5\psi_R + \bar{\psi}_R\gamma^5\partial_5\psi_L = 0,$$

because of the chirality projectors. This cancellation mechanism, that impedes any contribution from $i\partial_5\gamma^5$ to the fermion sector, is exactly the same one that annuls the interaction terms between the heavy Higgs and the fermions.

To study the effect of the new covariant derivative in the bosonic sector

$$\mathcal{L}_B = \frac{1}{2g^2}\widetilde{\text{Tr}}\left(\frac{1}{8}\text{Tr}^2\hat{D}^2 - \frac{1}{2}\text{Tr}\hat{D}^4\right), \quad (72)$$

we first calculate the explicit form of \hat{D}^2 :

$$\begin{aligned} \hat{D}^2 &= \partial^2 + \varphi^2 + 2A \cdot \partial + i\{\varphi, \partial_5\} - \partial_5^2 \\ &\quad + \mathcal{A}^2 + (\partial\mathcal{A}) + (\partial\Phi) + \{\mathcal{A}, \Phi\} + i(\partial_5\mathcal{A}). \end{aligned}$$

From here we can immediately conclude taking the trace that

$$\frac{1}{4}\text{Tr}\hat{D}^2 = \partial^2 + \varphi^2 + 2A \cdot \partial + i\{\varphi, \partial_5\} - \partial_5^2 + A^2 + \partial A.$$

There are several terms involved in each of the two trace expressions between parenthesis in (72), but a scrutiny of the possible types of combinations of products of γ^μ 's and γ^5 's shows that there is going to be much cancellation of terms between those

two trace expressions. With this in mind let us look at the following subtraction:

$$\begin{aligned}
\frac{1}{4^2} \text{Tr}^2 \hat{D}^2 - \frac{1}{4} \hat{D}^4 &= ((\partial \cdot A) + A^2)^2 - \frac{1}{4} \text{Tr}((\partial \mathcal{A}) + \mathcal{A}\mathcal{A})^2 \\
&\quad - \frac{1}{4} \text{Tr}((\partial \Phi)(\partial \Phi) + (\partial \Phi)\{\mathcal{A}, \Phi\} + (\partial \Phi)(i\gamma^5 \partial_5 \mathcal{A}) \\
&\quad + \{\mathcal{A}, \Phi\}(\partial \Phi) + \{\mathcal{A}, \Phi\}\{\mathcal{A}, \Phi\} + \{\mathcal{A}, \Phi\}(i\gamma^5 \partial_5 \mathcal{A}) \\
&\quad + (i\gamma^5 \partial_5 \mathcal{A})(\partial \Phi) + (i\gamma^5 \partial_5 \mathcal{A})\{\mathcal{A}, \Phi\} + (i\gamma^5 \partial_5 \mathcal{A})(i\gamma^5 \partial_5 \mathcal{A})) \\
&= \frac{1}{2} ((\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu])^2 + ((\partial_\mu \varphi) - i(\partial_5 A_\mu) + [A^\mu, \varphi])^2).
\end{aligned}$$

So we finally find the explicit form of the boson kinetic energy:

$$\mathcal{L}_B = \frac{1}{2g^2} \widetilde{\text{Tr}} \left((\partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu])^2 \right) + \frac{1}{g^2} \widetilde{\text{Tr}} \left((\partial_\mu \varphi - i\partial_5 A_\mu + [A_\mu, \varphi])^2 \right) \quad (73)$$

We see that the only change in the bosonic sector due to the extra dimension has been the addition of a new negative term in the kinetic energy of the scalar bosons, the term is $-i(\partial_5 A_\mu)$.

8.4 Five-dimensional formulation of the lagrangian.

With the help of definitions

$$A_5 \equiv -i\varphi, \quad \hat{\gamma}^5 \equiv i\gamma^5, \quad \hat{\gamma}^\mu \equiv \gamma^\mu, \quad (74)$$

the covariant derivative in five dimensions can be written in the interesting form

$$\begin{aligned}
\hat{D} &= \partial + i\gamma^5 \partial_5 + \mathcal{A} + \gamma^5 \varphi \\
&= \partial + \hat{\gamma}^5 \partial_5 + \mathcal{A} + \hat{\gamma}^5 A_5 \\
&= \hat{\gamma}^m (\partial_m + A_m), \quad m = 0, 1, 2, 3, 5,
\end{aligned}$$

where, as usual, a repeated index implies sum over its values. The expression that appears in the boson term of a GYMT in five dimensions can be similarly modified:

$$(\partial_\mu \varphi) - i(\partial_5 A_\mu) + [A_\mu, \varphi]^2 = i(\partial_{[\mu} A_{5]}) + i[A_\mu, A_5],$$

so that (73) takes the brief and convenient appearance

$$\mathcal{L}_B = \frac{1}{2g^2} \widetilde{\text{Tr}} \left((\partial_{[m} A_{n]} + [A_m, A_n]) (\partial^{[m} A^{n]} + [A^m, A^n]) \right), \quad m, n = 0, 1, 2, 3, 5,$$

where we have used $A^5 = -A_5$ and $\partial_5 = -\partial^5$, as we should if we are to be consistent with (64). Thus the boson term of a GYMT with an extra dimension looks superficially like a YMT in five dimensions. If we ignore completely the gauge structure of the theory, it would be a YMT, at least in the boson sector. Besides this difference there is another between these two types of theories. It hinges upon the fact that in

a YMT in a higher dimension it is necessary to consider the spinorial representations available to that dimension, while for the variation of a GYMT we have been working on there is no need for that, since the extra Dirac matrix $\hat{\gamma}^5$ we have been using is just an antihermitian form of the chirality matrix γ^5 of the original Dirac four-dimensional matrix algebra.

The five-dimensional covariant derivative (65) can also be written, using the new symbol $\hat{D}_m = \partial_m + A_m$, $m = 0, 1, 2, 3, 5$, in the familiar-looking form

$$\hat{D} = \hat{\gamma}^m \hat{D}_m.$$

The gauge transformation is modified into the five-dimensional form

$$\hat{\gamma}^m A_m \rightarrow U \hat{\gamma}^m A_m U^{-1} - (\hat{\gamma}^m \partial_m U) U^{-1}, \quad (75)$$

or

$$A_m \rightarrow U A_m U^{-1} - (\partial_m U) U^{-1}.$$

Furthermore, using the new symbol \hat{D}_m the boson term can be written simply as

$$\mathcal{L}_B = \frac{1}{2g^2} \widetilde{\text{Tr}} \left(([\hat{D}_m, \hat{D}_n])^2 \right).$$

8.5 Effect of the extra partial derivative ∂_5 .

To find what is the effect of the extra partial derivative we must find the equation of motion of the gauge fields. To this end we take the first-order variation of the bosonic term of the lagrangian with respect to δA_m (which includes now the scalar fields with the label A_5), with the result

$$\begin{aligned} \delta \mathcal{L}_B &= \frac{1}{g^2} \widetilde{\text{Tr}} \left(F^{mn} \left(\frac{\partial(\partial_{[m} A_{n]})}{\partial A_p} + \frac{\partial[A_m, A_n]}{\partial A_p} \right) \delta A_p \right) \\ &= \frac{2}{g^2} \widetilde{\text{Tr}} \left(F^{mn} (-\overleftrightarrow{\partial}_m \delta_n^p \delta A_p + \delta A_p \delta_m^p A_n + A_m \delta_n^p \delta A_p) \right) \\ &= -\frac{2}{g^2} \widetilde{\text{Tr}} ((\partial_m F^{mp} - A_n F^{pn} - F^{mp} A_m) \delta A_p) \\ &= -\frac{2}{g^2} \widetilde{\text{Tr}} ((\partial_m F^{mn} + A_m F^{mn} - F^{mn} A_m) \delta A_n) \\ &= -\frac{2}{g^2} \widetilde{\text{Tr}} (\partial_m F^{mn} + [A_m, F^{mn}]) \delta A_n = 0, \end{aligned} \quad (76)$$

where

$$F_{mn} \equiv \partial_{[m} A_{n]} + [A_m, A_n] = [\hat{D}_m, \hat{D}_n]. \quad (77)$$

The equations of motion are then

$$\partial_m F^{mn} + [A_m, F^{mn}] = 0.$$

Again, these equations look like YMT equations of motion, but they are not, due to the differing gauge structure. Of course, there is the additional difference of having $m = 5$ be a compactified dimension.

8.6 Ideas about the five-dimensional equations of motion.

The nonlinear character of equations (77) make their solution difficult. Furthermore other technical problems are involved in this problem, due to the compactification of one of the dimensions. The natural hope is that the least energy solution of these equations (or one of the least energy solutions), can result in a VEV of the order of the mass of the grand unification leptoquarks, say 10^{15} GeV, for the scalar field Ω . This field is hypothetically trapped in a small circle with coordinate x^5 . Since we know that $1\hbar c \approx 200$ MeV-fm, it is possible to conclude that the radius of the circle formed by the compactified dimension is of the order of $\sim 2 \times 10^{-29}$ cm. Obviously we cannot notice motion on such a small scale. We shall not dwell further in this topic. It goes without saying that it is crucial that this last dimension does not change the correct phenomenology of the four-dimensional GYMT, otherwise it would be necessary to do without it. But it would be a shame, because the extra dimension adds both sense and balance to the mathematics involved. For example, the extra dimension gives a clear geometric meaning to the scalar fields, as was discussed in Section 8. It also gives a clearer meaning to the gauge transformation of the boson fields, equations (34) and (75).

9 Final comments and criticisms.

We have gone through some very suggestive information concerning the structure of GYMTs. It would seem that these theories have relevance to particle physics, judging from the way they allow esthetic presentations of the GWSM and GUTs. However, both their mathematical and physical situations require clarification. The mathematical structures involved are complicated, and several points are not quite clear. We have gone in some detail into the possibility of using an extra dimension in their presentation with very interesting results. However, the correct way to proceed is still uncertain. Possibly the extra dimension that is introduced into this kind of GYMT is compactified, supplying a large mass scale.

From a physical point of view, one complaint is that these theories do not present an explicit potential $V(\phi)$ that would explain the nonzero VEVs of the Higgs fields ϕ . In this they differ from the GWSM. On the other hand, this potential is pretty much *ad hoc* in that model. Another important point is that the physical implications of the additional gauge symmetry that GYMTs contain ($SU(6)$ for the GUT we studied, for example) have not been yet properly evaluated.

In five dimensions GYMTs look like a YMT, but this is just a superficial similarity, since the gauge structure is different. In this type of GYMT each scalar field is taken to be the fifth component of a vector, and it always has a generator of the Lie group assigned to it that is not the same one assigned to the other components of the generalized gauge field. On the other hand, in a YMT each component of a particular vector field is associated with the same generator of the Lie group.

An interesting feature of GYMTs is that, even if branching rules tell us that two

scalar fields have the correct quantum numbers to couple, they are still prevented to do so by the mathematics.

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